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STRESS TESTS: FROM ARTS TO SCIENCE

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Abstract

Stress tests with handpicked scenarios might misrepresent risks either because dangerous scenarios are not considered or because the scenarios considered are too implausible. Systematic search for the worst case within some set of plausible scenarios is introduced to overcome these two pitfalls. For arbitrary loss functions we determine explicitly the worst case scenario over Kullback-Leibler spheres of plausible scenarios. Practical implementations of this method do not require any numerical optimisation. The method is illustrated in a number of example applications: linear and quadratic portfolios, stressed credit default probabilities, stressed rating transition correlations.

Keywords: scenario analysis, worst case, risk measures, multiple priors, model risk, relative entropy, maximum entropy principle, exponential family, Levenberg-Marquardt algorithm

JEL codes: C18, C44, C60, G01, G32, M48
AMS classification: 62C20, 90B50, 91B30, 94A17

1 Introduction

Stress tests evaluate the consequences of supposedly adverse scenarios. Currently scenarios are picked by hand. This is an art. Like in any art, results are not objective. Subjectivity is welcome in the arts but not in risk management. Depending on the choice of scenarios, stress test results might misrepresent risks either because the really dangerous scenarios are not considered or because the scenarios considered are too implausible. Then it may happen that banks go bankrupt although they have recently passed stress tests. Here we present a systematic way to perform stress tests. This reduces substantially the subjectivity of stress tests—and increases their credibility.

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The purpose of stress tests is to complement statistical risk measurement in two ways. While statistical risk measurement asks what the probabilities of big losses are, stress testing asks which scenarios lead to big losses. Knowledge of the dangerous scenarios should then suggest action reducing risk if desired. Second, stress tests should address model risk. They should inform about the size of losses in adverse scenarios without relying naïvely on a specific model, which perhaps uses inappropriate risk factors, or which works with the wrong risk factor distribution. The two complementary approaches must not be in contradiction. Berkowitz [2000] argued that stress tests should not take place in a framework separate from risk measurement. Reference to some risk factor distribution allows for a quantification of the plausibility of stress scenarios.

A first step towards more objective stress tests was made by Studer [1997, 1999]. He proposed to perform stress tests *systematically.* Instead of considering just a few hand-picked scenarios Studer searches for the worst case scenario among a set of plausible pure scenarios. In this way one ensures that no plausible scenario is missed and that only scenarios of sufficient plausibility are considered. Studer works with pure scenarios, which are (or which can be translated into) simultaneous realisations of the risk factors. Denote a pure scenario by an element $\boldsymbol{r}$ of the sample space. Studer measures the plausibility of pure scenarios by their Mahalanobis distance from the expected scenario and searches for the worst case over the ellipsoid of pure scenarios whose Mahalanobis distance is smaller than some threshold. He quantifies the harm done in a pure scenario $\boldsymbol{r}$ by a loss funtion $L(\boldsymbol{r})$.[1]

Studer's method addresses the two pitfalls. But this approach has problems of its own. First, choosing a Mahalanobis ellipsoid as scenario set is natural only for elliptical risk factor distributions, like the normal or the Student t-distribution. It is not clear how to choose sets of plausible scenarios if the risk factor distribution is not elliptical. For example, how should systematic stress tests be performed in credit risk models with discrete ratings classes? Second, stress testing with pure scenarios does not address model risk because it assumes a fixed risk factor distribution. Third, the Mahalanobis distance as a plausibility measure reflects only the first two moments of the risk factor distribution. This is not in line with intuition.

[1]The loss function $L(\boldsymbol{r})$ characterises the portfolio. For stress testers it might seem a bold assumption to know explicitly the loss function $L(\boldsymbol{r})$. After all, stress testers often need days to evaluate a complex portfolio in a given scenario. On the other hand, all standard quantitative risk management frameworks do work with a loss function, see e.g. McNeil et al. [2005, Chpt 2.1]. This loss function might be misspecified, for example because modellers want to abstract from some seemingly minor risk factors or because they wrongly specify the dependence of the portfolio value on the risk factors. Specifying the wrong loss function is part of model risk (the other part being a misspecification of the risk factor distribution). This part of model risk cannot be addressed by stress tests, which evaluate the loss in different scenarios. Be this as it may, we assume the loss function L of the portfolio to be given.

A given extreme scenario should be more plausible if the risk factor distribution has fatter tails. Fourth, the Mahalanobis distance depends on the choice of coordinates, as pointed out in Breuer [2008]. Fifth, the worst case loss over the ellipsoid is not a law-invariant risk measure: Portfolios might have the same profit loss distribution without having the same worst case loss.

We show a systematic way to perform stress tests avoiding the two pitfalls of stress testing with hand picked scenarios but overcoming the disadvantages of Studer's method. We propose to search systematically for the worst among the plausible mixed scenarios. We work with *mixed scenarios*, which are distributions of pure scenarios.[2] Mixed scenarios can be interpreted in different ways. First, as smeared versions of pure scenarios. Second, as risk factor distribution alternative to the reference distribution. A manager might hold 'views' about the risk factor distribution which for some reason are not reflected in the historical data, see Black and Litterman [1992] and Meucci [2009]. Third, mixed scenarios are routinely used when the portfolio depends on many, often thousands of risk factors. Then practitioners usually consider scenarios which specify only a few selected risk factors and therefore are easy to communicate. The values of the other risk factors are not fixed but assumed to have the conditional distribution given the values of the fixed risk factors; thus, in effect, mixed scenarios are considered.

As a natural measure of how bad is a mixed scenario represented by a risk factor distribution Q, we take the expected loss with respect to Q. Further, we measure the plausiblity of this scenario by the relative entropy of Q with respect to the estimated distribution ν, also called Kullback-Leibler distance or I-divergence, denoted by $D(Q||\nu)$. Regarding as admissible those scenarios for which this relative entropy does not exceed some threshold k, we formulate the systematic stress testing problem as a worst case search problem

$$\sup_{Q:D(Q||\nu)\leq k} \mathbb{E}_Q(L) =: \mathrm{MaxLoss}(L,k). \tag{1}$$

This problem was solved explicitly by Breuer and Csiszár [2010]. In Section 3 we will report this solution for the non-pathological cases.[3]

[2]This terminology is in analogy with game theory, which uses mixed strategies along with pure strategies, or with physics, which uses mixed states along with pure states. Sometimes the term generalised scenario is used for mixed scenarios, see Delbaen [2002].

[3]Observe that this problem is 'dual' to a problem of maximum entropy inference. If an unknown distribution Q had to be inferred when the available information specified only a feasible set of distributions, and a distribution ν were given as a prior guess of Q, the maximum entropy principle would suggest to infer the feasible distribution Q which minimizes $D(Q||\nu)$. (This name refers to the special case when the prior guess ν is the uniform distribution; then minimising $D(Q||\nu)$ is equivalent to maximising the entropy of Q.) In particular, if the feasible distributions were those with $\mathbb{E}_Q(L) = b$, for a constant b, we would arrive at the problem $\sup_{Q:\mathbb{E}_Q(L)=b} D(Q||\nu)$. Note that the objective function of the worst case problem (1) is the constraint in the maximum entropy problem (2), and

Systematic stress testing with mixed scenarios, as presented here, overcomes the five shortcomings of Studer's systematic stress test with pure scenarios. First, the scenario set $\{Q : D(Q||\nu) \leq k\}$ is defined not just for elliptical but for arbitrary reference risk factor distributions ν. As examples, systematic stress tests of credit ratings, transition probabilities and default correlations are performed in Sections 6 and 7. Second, stress testing with mixed scenarios addresses model risk and uncertainty aversion, as we argue in Section 2. Third, relative entropy as a measure of plausibility depends not just on the first two moments of estimated risk factor distribution ν. Fourth, MaxLoss as defined by (1) is invariant under coordinate transformations due to the invariance of D (see Kullback [1959, Corrolary 4.1, Chap. 2.4]) and of the integral defining expected loss. Fifth, MaxLoss over the Kullback-Leibler sphere is a law-invariant risk measure.

The paper is structured as follows. In Section 2 we briefly discuss related literature. Section 3 reports the solution to the worst case Problem (1) under constraints on relative entropy. In Section 4 we see that for linear portfolios and normal risk factor distributions $\text{MaxLoss}(L, k)$ and the Studer's worst case loss over the ellipsoid are equal. Section 5 gives explicit formulas for MaxLoss and the worst case scenario in the case of quadratic portfolios. In Section 6 we apply our theory to systematic stress tests of default probabilities, in Sections 7 and 8 to systematic stress tests of default correlations. These applications involve non-normal risk factor distributions and are therefore not accessible to systematic stress tests with pure scenarios.

2 Relation to the literature

Stress tests During the recent crisis supervisors subjected financial institutions to stress tests. These tests examined the capital needs of institutions in economic downturn scenarios with increased credit losses. Stress testing started in market risk, see e.g. Basel Committee on Banking Supervision [1996], but in recent years it has been applied to credit risk and macro analysis as well. A brief introduction into macro stress testing as well as an overview of EU country-level macro stress testing practices is given in a special feature of the Financial Stability Report of the European Central Bank [2006] or in Quagliariello [2009]. A detailed introduction into the topic and an overview of related literature is given in Sorge [2004]. In many countries, central banks' endeavour with macro stress testing was boosted by the IMF running a Financial Sector Assessment Program (FSAP). For details see Blaschke et al. [2001], Čihák [2004, 2007] and Jones et al. [2004]. A stress analysis of sector concentration risk in credit portfolios is given in Bonti et al. [2005]. Principles of sound stress testing practices have been laid down by the Basel Committee on Banking Supervision [2009]. All the

vice versa.

approaches mentioned above use hand-picked pure scenarios. The plausibility of scenarios is not always quantified and it remains unclear whether there are more severe scenarios of similar plausibility.

Relative entropy and other measures of plausibility It is natural to measure the plausibility of a mixed scenario Q by its distance from the estimated distribution ν. In the literature, various distances[4] of probability distributions are used. One family of such distances, the f-divergences of Csiszár [1963], Ali and Silvey [1966], and Csiszár [1967], correspond to convex funtions f on the positive numbers. Relative entropy corresponds to $f(t) = t \log t$, several other choices of f also give distances often used in statistics. For details about f-divergences see Liese and Vajda [1987].

From the range of possible distances we have chosen relative entropy, which appears the most versatile one with many applications in statistics, information theory, statistical physics, see e.g. Kullback [1959], Csiszár and Körner [1981], Cover and Thomas [2006], Jaynes [1968, 1982]. Relative entropy has already been used in econometrics, see Golan et al. [1996]. Using it also in the context of stress testing looks certainly reasonable, though we do not claim that among the various distances of distributions this one is necessarily the best for this purpose.

Let us note that in the context of inference the method of maximum entropy is distinguished by axiomatic considerations. Shore and Johnson [1980], Paris and Vencovská [1990], Jones and Byrne [1990] and Csiszár [1991] showed that it is the only method that satisfies certain intuitively desirable postulates. Still, as Uffink [1995, 1996] argued, relative entropy cannot be singled out as providing the only reasonable method of inference. Csiszár [1991] determined what alternatives come into account if some postulates are relaxed. Grunwald and Dawid [2004] argue that distances between distributions might be chosen in a utility dependent way. Relative entropy is natural only for decision makers with logarithmic utility. Picking up this idea, for decision makers with non-logarithmic utility one might define the radius of the scenario set in terms of some utility dependent distance. But this is not the approach of this paper. In our framework utility may enter into the loss function L but not into the scenario set.

3 Maximum Loss under Relative Entropy Constraints

We now report from Breuer and Csiszár [2010] the solution of the worst case problem (1) involved in systematic stress testing with mixed scenarios. The

[4] Distance is meant in a broad sense, requiring neither symmetry nor the triangle inequality; those properties of distances in the narrow sense do not hold even for relative entropy.

solution relies on techniques familiar in the theory of exponential families, see Barndorff-Nielsen [1978], and large deviations theory, see Dembo and Zeitouni [1998]. A key tool is the function

$$\Lambda(\theta, L) := \log \left(\int e^{\theta L(\boldsymbol{r})} d\nu(\boldsymbol{r}) \right), \tag{2}$$

where θ is a positive real number. If the loss function L is clear from the context, we will simply write $\Lambda(\theta)$.

The relative entropy of a probability distributions Q with respect to a reference distribution ν is defined as

$$D(Q||\nu) := \begin{cases} \int \log \frac{dQ}{d\nu}(\boldsymbol{r}) dQ(\boldsymbol{r}) & \text{if } Q \ll \nu \\ +\infty & \text{if } Q \not\ll \nu \end{cases}$$

where $Q \ll \nu$ denotes absolute continuity of the distribution Q with respect to the distribution ν.

The solution to the worst case problem (1) generically looks as follows. The generic case obtains when the following assumptions are satisfied: (i) If $\operatorname{ess\,sup}(L)$ is finite, assume k is smaller than $k_{\max} := -\log(\nu(\{\boldsymbol{r} : L(\boldsymbol{r}) = \operatorname{ess\,sup}(L)\}))$. (ii) Assume $\theta_{\max} := \sup\{\theta : \Lambda(\theta) < +\infty\} > 0$, (iii) If $\theta_{\max}, \Lambda(\theta_{\max})$, and $\Lambda'(\theta_{\max})$ are all finite, assume k does not exceed $k_{\max} := \theta_{\max}\Lambda'(\theta_{\max}) - \Lambda(\theta_{\max})$. Breuer and Csiszár [2010] also give solutions to the problem in the pathological cases where one or more of the three assumptions are violated. But this is not needed for the present purpose. Under assumptions (i)-(iii) the equation

$$\theta\Lambda'(\theta) - \Lambda(\theta) = k \tag{3}$$

has a unique positive solution $\overline{\theta}$. The worst case scenario $\overline{Q}$ is the distribution with ν-density

$$\frac{d\overline{Q}}{d\nu}(\boldsymbol{r}) := \frac{e^{\overline{\theta}L(\boldsymbol{r})}}{\int e^{\overline{\theta}L(\boldsymbol{r})} d\nu(\boldsymbol{r})} = e^{\overline{\theta}L(\boldsymbol{r}) - \Lambda(\overline{\theta})}. \tag{4}$$

The Maximum Loss achieved in the worst case scenario $\overline{Q}$ is

$$\text{MaxLoss}(L, k) = \Lambda'(\overline{\theta}). \tag{5}$$

This result provides a practical procedure for calculating MaxLoss in the generic case, which is illustrated in Fig. 1:

1. Calculate $\Lambda(\theta)$ from (2). This involves the evaluation of an n-dimensional integral.

2. Starting from the point $(0, -k)$, lay a tangent to the curve $\Lambda(\theta)$.

3. MaxLoss is given by the slope of the tangent.

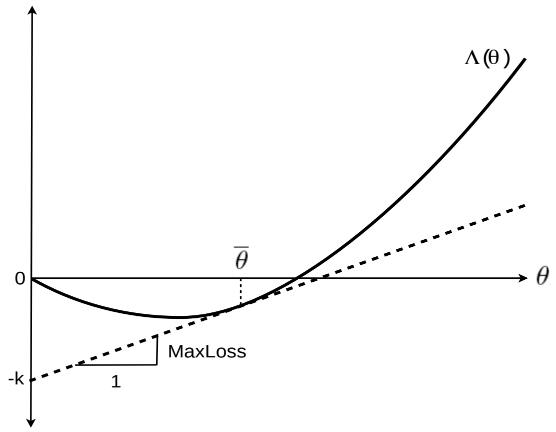

Figure 1: **Calculation of MaxLoss from Λ.** MaxLoss is the slope of the tangent to the curve $\Lambda(\theta)$ passing through $(0, -k)$. $\overline{\theta}$ is the θ-coordinate of the tangent point.

4. The worst case scenario is the distribution with density (4), where $\overline{\theta}$ is the θ-coordinate of the tangent point.

How should one choose the radius k of the Kullback-Leibler sphere? k is a parameter in Problem (1), in the same way as the confidence level is a parameter for value at risk or expected shortfall. Which choice of k is sensible? MaxLoss(k) dominates Tail-VaR at the level $\exp(-k)$:

$$\sup_{A:\nu(A)\geq e^{-k}} \int_A L(\boldsymbol{r})d\nu(\boldsymbol{r})/\nu(A) \leq \sup_{Q:D(Q||\nu)\leq k} \mathbb{E}_Q(L).$$

(This is true because the distribution Q_A with density $dQ_A/d\nu := 1_A/\nu(A)$ satisfies $D(Q_A||\nu) \leq k$ if $\nu(A) \geq \exp(-k)$.) This inequality suggests reasonable orders of magnitude for k. For a 1%-tail the corresponding k is $-\log(0.01) = 4.6$. An alternative way to choose k would be to take k-values realised in historical crisis.

4 Linear Portfolios

Studer [1997, 1999] proposed to search systematically for the worst pure scenarios within the ellipsoid Ell_h of pure scenarios whose Mahalanobis distance $\text{Maha}(\boldsymbol{r}) := \sqrt{(\boldsymbol{r} - \mathbb{E}(\boldsymbol{r}))^T \Sigma^{-1} (\boldsymbol{r} - \mathbb{E}(\boldsymbol{r}))}$ from the expected scenario is smaller than some threshold h. Here h is either determined so as to give the ellipsoid some desired probability mass as by Studer [1997, 1999] or independent of the number of risk factors as by Breuer [2008], thereby avoiding the problem of dimensional dependence.

Let us assume the loss in portfolio value is given by a linear function of one risk factor, $L(r) = (\mu - r)l$, which is normally distributed, $r \sim \nu = N(\mu, \sigma^2)$. In this case the ellipsoid is simply an interval, $\mathrm{Ell}_h = [\mu - h\sigma, \mu + h\sigma]$. Studer's method yields as worst case scenario a move of h standard deviations up or down, depending on whether l is positive or negative. The worst case loss is $h\sigma|l|$. The next proposition shows that in this special case of a linear portfolio depending on one normally distributed risk factor, Studer's method leads to the same result as our method.

Proposition 1. *Assume the loss in portfolio value is given by a linear function of one risk factor, $L(r) = (\mu - r)l$, which is normally distributed, $r \sim \nu = N(\mu, \sigma^2)$. The worst case scenario $\overline{Q}$ is a normal distribution with the same variance σ^2 as the reference risk factor distribution ν, but with mean equal to*

$$\mu + h\sigma \,\mathrm{sgn}(l),$$

where $h = \sqrt{2k}$. This mean equals the worst pure scenario over $\mathrm{Ell}_h = [\mu - h\sigma, \mu + h\sigma]$. The maximum expected loss is

$$\mathbb{E}_{\overline{Q}}(L) = h\sigma|l|,$$

which equals the loss in the worst pure scenario over Ell_h.

The same holds if the portfolio loss is given by a linear function of n risk factors, $L(\boldsymbol{r}) = \boldsymbol{l} \cdot (\boldsymbol{\mu} - \boldsymbol{r})$, which are normally distributed with mean $\boldsymbol{\mu}$ and covariance matrix Σ, $\boldsymbol{r} \sim \nu = N(\boldsymbol{\mu}, \Sigma)$:

Proposition 2. *The worst case scenario is a normal distribution with the same covariance matrix Σ as the reference distribution ν, but with mean equal to*

$$\boldsymbol{\mu} - \frac{h}{\sqrt{\boldsymbol{l}^T \Sigma \boldsymbol{l}}} \Sigma \boldsymbol{l},$$

where $h = \sqrt{2k}$. This mean equals the worst pure scenario over Ell_h. The worst case loss is

$$\mathrm{MaxLoss}(L, k) = \sqrt{2k}\sqrt{\boldsymbol{l}^T \Sigma \boldsymbol{l}},$$

which equals the loss in the worst pure scenario over Ell_h.

5 Quadratic Portfolios

For normal reference distribution ν, when the loss function is not linear but quadratic the worst case scenario still is a normal, but its covariance is no longer the same as the covariance of the reference distribution ν. Let us assume the loss in portfolio value is given by a quadratic function

$$L(\boldsymbol{r}) = -\frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{r})^T G(\boldsymbol{\mu} - \boldsymbol{r}) - \boldsymbol{l} \cdot (\boldsymbol{\mu} - \boldsymbol{r})$$

of n risk factors which are normally distributed, $\boldsymbol{r} \sim \nu = N(\boldsymbol{\mu}, \Sigma)$. Here G is a quadratic not necessarily positive definite matrix and $\boldsymbol{l}$ is a vector representing a linear portfolio component. Denote by U the Cholesky decomposition of the covariance matrix $\Sigma = U^T U$ and by γ_i the eigenvalues of UGU^T.

Proposition 3. *If all eigenvalues γ_i of UGU^T are positive, then*

$$S(\theta) := (\theta UGU^T + \mathbf{1})^{-1}$$

exists for all positive θ. If the smallest eigenvalue of UGU^T is negative, $S(\theta)$ exists for all θ in the interval $[0, -1/\min(\gamma_i))$. There is a unique positive $\overline{\theta}$ which solves the equation

$$\frac{1}{2}\left[\theta^2 \boldsymbol{l}^T U^T S(\theta)^2 U\boldsymbol{l} + \log n - \sum_i \left(\frac{\theta\gamma_i}{1+\theta\gamma_i} - \log(1+\theta\gamma_i)\right)\right] = k. \qquad (6)$$

The worst case scenario $\overline{Q}$ is a normal distribution with covariance matrix

$$U^T S(\overline{\theta}) U$$

and mean

$$\boldsymbol{\mu} + \overline{\theta} U^T S(\overline{\theta}) U\boldsymbol{l}.$$

The maximum expected loss is

$$\mathrm{MaxLoss}(L,k) = \frac{\overline{\theta}}{2}\boldsymbol{l}^T U^T \left(S(\overline{\theta}) + S(\overline{\theta})^2\right) U\boldsymbol{l} - \frac{1}{2}\sum_i \frac{\gamma_i}{1+\overline{\theta}\gamma_i}.$$

This proposition allows to determine the worst case scenario for quadratic portfolios in an efficient way. It generalises the MaxLoss algorithm for quadratic portfolios of Studer [1997, Section 3.4] from pure scenarios to mixed scenarios. It can be regarded as a generalisation of the Levenberg-Marquardt algorithm (see Fletcher [1987, p.101]), which traditionally is used to calculate numerically the global minimum of a quadratic function in a ball. For a linear portfolio $G = 0$ and Proposition 3 reduces to Proposition 2.

6 Stressed default probabilities

Systematic credit risk stress testing with pure scenarios can be performed for credit risk models where the loss is a function of normally distributed risk factors, see Breuer et al. [2009]. Something similar could be done for the asymptotic single factor models underlying the capital rules of the Basel Committee on Banking Supervision [2005, p. 64], which use a normally distributed latent risk factor.

But for credit risk models with irreducibly discrete risk factor distributions, such as RiskMetrics or CreditRisk+, elliptic sets of pure scenarios do not make sense. For this reason systematic stress tests with pure scenarios cannot be applied to such models. By ways of a simple example we show how such credit risk models can be submitted to systematic stress tests with mixed scenarios.

Consider an obligor who at some future time can be in one of n rating classes (states). The probabilities of a transition from the current rating class to some rating i is p_i. The vector $\nu = (p_1, \ldots, p_n)$ of transition probabilities plays the role of the reference risk factor distribution ν. Market data and obligor data specify for each possible final rating the loss l_i caused by a transition into that class. Systematic stress testing is the problem to find the transition probabilities $\overline{\boldsymbol{p}}$ which maximise the expected loss $\overline{\boldsymbol{p}} \cdot \boldsymbol{l}$ and whose relative entropy with respect to the reference transition probabilities ν is smaller than k.

We get

$$\Lambda(\theta) = \log \left(\sum_{j=1}^{n} p_j \exp(\theta l_j) \right)$$

and one calculates $\Lambda'(\theta) = \exp(-\Lambda(\theta)) \sum_{j=1}^{n} p_j l_j \exp(\theta l_j)$. $\overline{\theta}$ is determined numerically from the equation $\theta \Lambda'(\theta) - \Lambda(\theta) = k$. This equation has a solution if and only if $k < k_{\max} = -\log p_1$, where p_1 is the probability of the worst rating class 'default'. Beyond $k_{\max}$ the maximum expected loss equals the supremum of the loss function, which equals loss given default, l_1, because the obligor defaults with certainty.

The vector of worst case transition probabilities is

$$\overline{p}_i = \frac{\exp(\overline{\theta} l_i)}{\sum_{j=1}^{n} p_j \exp(\overline{\theta} l_j)} p_i.$$

Compare this to the reference transition probabilities p_i. Into rating classes better than the original rating, the worst case transition probability is lower than the estimated transition probability. Into worse rating classes the worst case transition probabilities are higher.

Table 1 gives a numerical example. Consider a bond of rating A. Over a time period of one year its rating can migrate into the rating classes with the estimated probabilities given in the second row of the table. This causes a loss which is given in the first row. (These loss numbers were determined from credit spreads of A-rated industrial bonds maturing in 5 years, as given by Bloomberg.) The last line gives the worst case transition probabilities at a plausibility level of $k = 2$. Downgradings have a higher probability, upgradings a lower. Under the estimated transition probabilities the expected loss is 0.37% of the bond value. Under the worst case transition probabilities the expected loss is 19.07% of the bond value.

Table 1: Stressed transition probabilities

	AA1-2	AA3	A	BBB	BB	Default
loss from transitions [%]	-3.20%	-1.07%	0.00%	3.75%	15.83%	51.80%
est'd trans. prob. [%]	0.09	2.60	90.75	5.50	1.00	0.06
worst c. trans. prob. [%]	0.036	1.34	53.53	5.37	4.91	34.8

7 Stressed Default Correlations

In portfolio credit risk models the dependence between defaults of individual names is of central importance. In most models the dependence between defaults arises from the dependence between asset values or from the dependence of defaults on some common factors. For a comparative analysis of portfolio credit risk models see Gordy [2000]. As a toy example consider a simplified firm's value model of a two name loan portfolio, as in Schönbucher [2001, Section 4.1].

The default of obligor i is triggered by the passage of her asset value below some default threshold K_i. The asset value of firm i is denoted by r_i. Assume that r_i is normally distributed. Without loss of generality we set the initial value to zero, $r_i(0) = 0$, and standardise their development such that $r_i \sim N(0, 1)$. The loss function is

$$L(\boldsymbol{r}) = \sum_{i=1}^{n} l_i 1_{(-\infty, K_i]}(r_i),$$

where l_i is the loss given default of obligor i. The default thresholds K_i are calibrated to produce the single name default probabilities p_i, $K_i = \Phi^{-1}(p_i)$. The asset value of different obligors are correlated with each other. In case of 2 obligors the only free parameter is their asset correlation ρ. Estimation errors of ρ have no influence on the expected loss of the portfolio. But ρ is relevant for the probability of both obligors defaulting simultaneously, $p_{AB} := \Phi_{2,\rho}(K_1, K_2)$ and thus for the tails of the loss distribution. (Here $\Phi_{2,\rho}$ is the cumulative distribution function of the two-dimensional normal with means zero, unit variances and correlation ρ). The Λ-function is given by

$$\begin{aligned} \Lambda(\theta) \quad = \quad & \log \Big((p_A - p_{AB}) e^{\theta l_A} + (p_B - p_{AB}) e^{\theta l_B} \\ & + p_{AB} e^{\theta (l_A + l_B)} + (1 - p_A - p_B + p_{AB}) \Big) . \end{aligned}$$

$\overline{\theta}$ is determined numerically from (3), which has a unique solution if and

Table 2: Stressed default correlations: Two obligors with LGD=0.4 resp. 0.5. Two rating classes. k=2, $\rho = 0.5$.

	no def.	just A def.	just B def	both def	def. corr.	exp. loss
est'd dist.	98.66%	1.32%	0.013%	0.007%	4.23%	0.67%
worst c. dist.	43.02%	47.94%	0.19%	8.85%	26.15%	32.01%

only if $k < k_{\max} = -\log p_{AB}$. The derivative at $\overline{\theta}$ determines MaxLoss,

$$\text{MaxLoss}(k) = \frac{(p_A - p_{AB}) l_A e^{\overline{\theta} l_A} + (p_B - p_{AB}) l_B e^{\overline{\theta} l_B} + p_{AB}(l_A + l_B) e^{\overline{\theta}(l_A + l_B)}}{(p_A - p_{AB}) e^{\overline{\theta} l_A} + (p_B - p_{AB}) e^{\overline{\theta} l_B} + p_{AB} e^{\overline{\theta}(l_A + l_B)} + (1 - p_A - p_B + p_{AB})}.$$

The worst case asset value density $d\overline{Q}/d\nu$ is not a normal. The probability it puts on the regions of both obligors defaulting, just obligor A defaulting, just obligor B defaulting, and no obligor defaulting are

$$\begin{aligned}
\overline{p}_{AB} &= \overline{Q}((-\infty, K_1] \times (\infty, K_2]) = \exp[\overline{\theta}(l_A + l_B) - \Lambda(\overline{\theta})] p_{AB} \\
\overline{p}_A - \overline{p}_{AB} &= \overline{Q}((-\infty, K_1] \times (K_2, \infty]) = \exp[\overline{\theta} l_A - \Lambda(\overline{\theta})] p_A - \overline{p}_{AB} \\
\overline{p}_B - \overline{p}_{AB} &= \overline{Q}((K_1, \infty] \times (-\infty, K_2]) = \exp[\overline{\theta} l_B - \Lambda(\overline{\theta})] p_B - \overline{p}_{AB} \\
\overline{p}_0 &= \overline{Q}((K_1, \infty] \times (K_2, \infty]) = 1 - \overline{p}_A - \overline{p}_B + \overline{p}_{AB}
\end{aligned}$$

A numerical example is given in Table 2. For single name default probabilities of $p_A = 1.33\%$ and $p_B = 0.02\%$, asset correlation $\rho = 0.5$, and losses given default of $l_A = 0.5$ and $l_B = 0.4$ we get the following results at $k = 2$. Under the stressed asset value distribution, the default correlation is 26.15% instead of 4.23%, leading to a stress expected loss of 32.01% instead of 0.67% under quiet circumstances.

8 Stressed transition correlations with general firm value distributions

The analysis can easily be generalised to more than two obligors and more than two rating classes, and to other asset value distributions than multivariate normals.

Let $j = 1, \ldots, n$ denote the n obligors and $i = 1, \ldots, I$ the rating classes. Rating class $i = 1$ is default, rating class $i = I$ is the best. Denote by ν the joint asset value distribution and by F_ν^j the cumulative distribution function of the j-th marginal. For each obligor j the probability to end up in rating class i is given, call it p_{ji}. The cumulative transition probability of ending up in a rating class worse or equal to i is $P_{ji} := \sum_{k \le i} p_{jk}$. The transition thresholds for obligor j are

$$K_{ji} := (F_\nu^j)^{-1}(P_{ji}).$$

Set $K_{j0} := (F_\nu^j)^{-1}(0)$. The asset value space is partitioned into cells

$$\Delta_{\boldsymbol{i}} := \{\boldsymbol{r} : r_1 \in (K_{1,i_1-1}, K_{1,i_1}], r_2 \in (K_{2,i_2-1}, K_{2,i_2}], \ldots, r_n \in (K_{n,i_n-1}, K_{n,i_n}]\}.$$

The vector $\boldsymbol{i}$ (bold face) describes the final state of the loan portfolio, giving the final ratings of all obligors: If $\boldsymbol{r} \in \Delta_{\boldsymbol{i}}$ obligor 1 ends up in class i_1, obligor 2 in class i_2, etc. Call

$$p_{\boldsymbol{i}} := \int_{\Delta_{\boldsymbol{i}}} d\nu(\boldsymbol{r})$$

the probability of having final ratings $\boldsymbol{i}$. Let l_{ji} be the loss or profit from obligor j ending up in class i. Then $l_{\boldsymbol{i}} := \sum_{k=1}^n l_{ki_k}$ is the portfolio loss if the obligors end up in classes $\boldsymbol{i}$. The portfolio loss function is therefore

$$L(\boldsymbol{r}) = \sum_{\text{cells } \boldsymbol{i}} l_{\boldsymbol{i}} 1_{\Delta_{\boldsymbol{i}}}(\boldsymbol{r}).$$

Now we can follow the procedure of Section 3. Λ is

$$\Lambda(\theta) = \log \left(\sum_{\text{cells } \boldsymbol{i}} e^{\theta l_{\boldsymbol{i}}} p_{\boldsymbol{i}} \right),$$

and its derivative is $\Lambda'(\theta) = e^{-\Lambda(\theta)} \sum_{\text{cells } \boldsymbol{i}} e^{\theta l_{\boldsymbol{i}}} l_{\boldsymbol{i}} p_{\boldsymbol{i}}$. $\overline{\theta}$ is determined numerically from (3), which has a unique solution if and only if $k < k_{\max} = -\log p_{\mathbf{1}}$. The worst case asset value distribution at level k is

$$\frac{d\overline{Q}}{d\nu}(\boldsymbol{r}) = e^{\overline{\theta} l_{\boldsymbol{i}(\boldsymbol{r})} - \Lambda(\overline{\theta})},$$

where $\boldsymbol{i}(\boldsymbol{r})$ is the cell of $\boldsymbol{r}$. The worst case probability of the final portfolio state $\boldsymbol{i}$ is

$$\overline{p}_{\boldsymbol{i}} = e^{\overline{\theta} l_{\boldsymbol{i}} - \Lambda(\overline{\theta})} p_{\boldsymbol{i}}.$$

9 Conclusion

Current stress tests in financial institutions use hand-picked scenarios, which might not be really dangerous or not sufficiently plausible. Systematic stress testing with pure scenarios is intended to overcome these two pitfalls but is rarely used in practice, partly because it is restricted to normal risk factor distributions, partly because of the computational demands. This paper introduces systematic stress testing for general distributions. The worst case scenarios identified with this method are plausible, severe, and suggest risk reducing action. An important practical achievement is the derivation of closed form expressions for the worst case scenario and the Maximum Loss. The new method does not require any numerical optimisation. This paves the way for widespread practical use.

The worst case distribution $\overline{Q}$ of eq. (4) is the Esscher transform of the estimated risk factor distribution ν, with parameter $\overline{\theta}$. The Esscher transform is a popular actuarial pricing measure. It gives rise to the unique no-arbitrage price in the incomplete market of a compound Poisson risk process, when the price is required to be an equilibrium price in a market of exponential utility maximisers (Embrechts [1997]). The Esscher price can now be given a new interpretation. It is the worst case expected loss among a Kullback-Leibler sphere of plausible distributions. The parameter θ of the Esscher transform is not arbitrary but determined via (3) by the radius of the Kullback-Leibler sphere.

Appendix

Proof of Proposition 1

Proof. First we get by quadratic completion

$$\begin{aligned} \theta L(\boldsymbol{r}) - \frac{1}{2\sigma^2}(\mu - r)^2 &= \theta l(\mu - r) - \frac{1}{2\sigma^2}(\mu - r)^2 \\ &= \frac{\theta^2\sigma^2 l^2}{2} - \frac{1}{2\sigma^2}(\mu' - r)^2 \end{aligned} \tag{7}$$

where $\mu' = \mu - \sigma^2\theta l$. With this one gets for $\Lambda(\theta)$:

$$\begin{aligned} \Lambda(\theta) &= \log\left(\int_{\mathbb{R}^n} e^{\theta L(\boldsymbol{r})} d\nu(r)\right) &(8) \\ &= \log\left(\frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} \exp\left(\theta L(\boldsymbol{r}) - \frac{1}{2\sigma^2}(\mu - r)^2\right) dr\right) \\ &= \theta^2\sigma^2 l^2/2. &(9) \end{aligned}$$

Now one calculates $\overline{\theta}$ as the positive solution of $\theta\Lambda'(\theta) - \Lambda(\theta) = k$ yielding

$$\overline{\theta} = \frac{\sqrt{2k}}{\sigma|l|} = \frac{h}{\sigma|l|}. \tag{10}$$

According to Theorem **??** the worst case loss is

$$\mathbb{E}_{\overline{Q}}(L) = \Lambda'(\overline{\theta}) = h\sigma|l|.$$

The worst case scenario is given by

$$\begin{aligned} d\overline{Q}(\boldsymbol{r}) &= e^{\overline{\theta}L(r) - \Lambda(\overline{\theta})} d\nu(r) \\ &= e^{-\Lambda(\overline{\theta})}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(\overline{\theta}L(\boldsymbol{r}) - \frac{1}{2\sigma^2}(\mu - r)^2\right) dr \\ &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(\mu' - r)^2\right), \end{aligned} \tag{11}$$

where we used eq. (7). This implies that the worst case scenario is a normal distribution with mean $\mu' = \mu - \sigma^2\overline{\theta}l = \mu - h\sigma\text{sgn}(l)$ and covariance σ^2. The square of the Mahalanobis distance between μ and the mean of the worst case scenario is

$$\text{Maha}(\mu, \mu')^2 = (\mu - \mu')^2/\sigma^2 = (\sigma^2\overline{\theta}l)^2/\sigma^2 = h^2,$$

because of (10). □

Proof of Proposition 2

Proof. Denote by U the orthogonal matrix of eigenvectors of Σ^{-1}. Then $U^T S^{-1} U = \Sigma^{-1}$, where S^{-1} is the diagonal matrix of eigenvalues of Σ^{-1}. Using the substitutions $\boldsymbol{y} := U(\boldsymbol{\mu} - \boldsymbol{r})$ and $\boldsymbol{z} = U\boldsymbol{l}$ we get

$$\begin{aligned}\theta L(\boldsymbol{r}) - \frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{r})^T\Sigma^{-1}(\boldsymbol{\mu}-\boldsymbol{r}) &= \theta \boldsymbol{l}\cdot(\boldsymbol{\mu}-\boldsymbol{r}) - \frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{r})^T U^T S^{-1} U(\boldsymbol{\mu}-\boldsymbol{r}) \\ &= \theta \boldsymbol{z}\cdot\boldsymbol{y} - \frac{1}{2}\boldsymbol{y}^T S^{-1}\boldsymbol{y}.\end{aligned}$$

By quadradic completion this yields

$$\theta L(\boldsymbol{r}) - \frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{r})^T\Sigma^{-1}(\boldsymbol{\mu}-\boldsymbol{r}) = \frac{\theta^2}{2}\boldsymbol{z}^T S\boldsymbol{z} - \frac{1}{2}(\boldsymbol{y}-\theta S\boldsymbol{z})^T S^{-1}(\boldsymbol{y}-\theta S\boldsymbol{z}). \quad (12)$$

With this one calculates Λ:

$$\begin{aligned}\Lambda(\theta) &= \log\left(\int_{\mathbb{R}^n} e^{\theta L(\boldsymbol{r})} d\nu(\boldsymbol{r})\right) \\ &= \log\left((2\pi)^{-n/2}|\Sigma|^{-1/2}\int_{\mathbb{R}^n}\exp\left(\theta L(\boldsymbol{r}) - \frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{r})^T\Sigma^{-1}(\boldsymbol{\mu}-\boldsymbol{r})\right)d\boldsymbol{r}\right) \\ &= \frac{\theta^2}{2}\boldsymbol{z}^T S\boldsymbol{z} =: \frac{\theta^2}{2}\alpha^2 \qquad (13)\end{aligned}$$

with $\alpha^2 := \boldsymbol{z}^T S\boldsymbol{z} = \boldsymbol{l}^T\Sigma\boldsymbol{l}$. Now one calculates $\overline{\theta}$ as the positive solution of $\theta\Lambda'(\theta) - \Lambda(\theta) = k$ yielding

$$\overline{\theta} = \frac{\sqrt{2k}}{\alpha} = \frac{h}{\alpha}.$$

According to Theorem **??** the worst case loss is

$$\mathbb{E}_{\overline{Q}}(L) = \Lambda'(\overline{\theta}) = h\alpha.$$

The worst case scenario is given by

$$\begin{aligned}d\overline{Q}(\boldsymbol{r}) &= e^{\theta L(\boldsymbol{r})-\Lambda(\theta)}d\nu(\boldsymbol{r}) \\ &= e^{\theta L(\boldsymbol{r})-k}(2\pi)^{-n/2}|\Sigma|^{-1/2}\int_{\mathbb{R}^n} e^{-\frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{r})^T\Sigma^{-1}(\boldsymbol{\mu}-\boldsymbol{r})}d\boldsymbol{r} \\ &= (2\pi)^{-n/2}|\Sigma|^{-1/2}\exp\left(-\frac{1}{2}(\boldsymbol{y}-\overline{\theta}S\boldsymbol{z})^T S^{-1}(\boldsymbol{y}-\overline{\theta}S\boldsymbol{z})\right)d\boldsymbol{r} \quad (14)\end{aligned}$$

where we used eq. (12). This implies that in y-coordinates the worst case scenario is a normal distribution with mean $\frac{h}{\alpha}S\boldsymbol{z}$ and covariance matrix S. In the original r-coordinates the worst case scenario is a normal with mean $\boldsymbol{\mu} - \frac{h}{\alpha}\Sigma\boldsymbol{l}$ and covariance Σ. The Mahalanobis distance between $\boldsymbol{\mu}$ and the mean of the worst case scenario is

$$\text{Maha}(\boldsymbol{\mu}, \boldsymbol{\mu} - \frac{h}{\alpha}\Sigma\boldsymbol{l})^2 = (-h\Sigma\boldsymbol{l}/\alpha)^T\Sigma^{-1}(-h\Sigma\boldsymbol{l}/\alpha) = (h^2/\alpha^2)\boldsymbol{l}^T\Sigma\boldsymbol{l} = h^2,$$

using $\alpha^2 := \boldsymbol{z}^T S\boldsymbol{z} = \boldsymbol{l}^T\Sigma\boldsymbol{l}$. □

Proof of Proposition 3

Proof. The Cholesky decomposition U of the covariance matrix $\Sigma = U^T U$ transforms the original $\boldsymbol{r}$-coordinates into new coordinates $\boldsymbol{y} := (U^{-1})^T(\boldsymbol{\mu} - \boldsymbol{r})$, which are standard normally distributed, $\boldsymbol{y} \sim N(\boldsymbol{0}, \boldsymbol{1})$. A distribution Q is in $S(\nu, k)$ if and only if the distribution $Q'(B') := Q(B)$ is in $S(N(\boldsymbol{0}, \boldsymbol{1}), k)$, where $B' := \{\boldsymbol{y} : \boldsymbol{y} = (U^{-1})^T(\boldsymbol{\mu} - \boldsymbol{r}), \boldsymbol{r} \in B\}$ for any $B \in \mathbb{F}$. The transformed loss function is $\overline{L}(\boldsymbol{y}) := -\frac{1}{2}\boldsymbol{y}^T G' \boldsymbol{y} - \boldsymbol{z} \cdot \boldsymbol{y}$, where $G' := UGU^T$ and $\boldsymbol{z} := U\boldsymbol{l}$. This ensures $L(\boldsymbol{r}) = \overline{L}(\boldsymbol{y})$. We will now establish that in $\boldsymbol{y}$-coordinates the worst case scenario is a normal with mean $-\overline{\theta}(\overline{\theta}G' + \boldsymbol{1})^{-1}\boldsymbol{z}$ and covariance matrix $(\overline{\theta}G' + \boldsymbol{1})^{-1}$, where $\overline{\theta}$ is the solution of (6). This entails the proposition.

First, quadratic completion yields

$$\begin{aligned}
\theta \overline{L}(\boldsymbol{y}) - \frac{1}{2}\boldsymbol{y}^T\boldsymbol{y} &= -\boldsymbol{y}^T(\theta G' + \boldsymbol{1})\boldsymbol{y} - \theta \boldsymbol{z} \cdot \boldsymbol{y} \\
&= -\boldsymbol{y}^T S(\theta)^{-1}\boldsymbol{y} - \theta \boldsymbol{z} \cdot \boldsymbol{y} \\
&= \frac{\theta^2}{2}\boldsymbol{z}^T S(\theta)\boldsymbol{z} - \frac{1}{2}(\boldsymbol{y} + \theta S\boldsymbol{z})^T S(\theta)^{-1}(\boldsymbol{y} + \theta S\boldsymbol{z}),
\end{aligned}$$

where we abbreviated $S(\theta) := (\theta G' + \boldsymbol{1})^{-1}$. (The inverse of $\theta G' + \boldsymbol{1}$ exists since either all eigenvalues γ_i are positive, or $\theta < -1/\min \gamma_i$ if some γ_i are negative.) With this one calculates $\Lambda(\theta)$:

$$\begin{aligned}
\Lambda(\theta) &= \log\left(\int_{\mathbb{R}^n} e^{\theta L(\boldsymbol{r})} d\nu(\boldsymbol{r})\right) \\
&= \log\Big((2\pi)^{-n/2}|\boldsymbol{1}|^{-1/2} \qquad (15) \\
&\qquad \int_{\mathbb{R}^n} \exp\left(\frac{\theta^2}{2}\boldsymbol{z}^T S(\theta)\boldsymbol{z} - \frac{1}{2}(\boldsymbol{y} + \theta S\boldsymbol{z})^T S(\theta)^{-1}(\boldsymbol{y} + \theta S\boldsymbol{z})\right) d\boldsymbol{y}\Big) \\
&= \frac{\theta^2}{2}\boldsymbol{z}^T S(\theta)\boldsymbol{z} + \frac{1}{2}\log|S(\theta)| - \frac{1}{2}\log n
\end{aligned}$$

Therefore

$$\Lambda'(\theta) = \theta \boldsymbol{z}^T S(\theta)\boldsymbol{z} + \frac{\theta^2}{2}\boldsymbol{z}^T \frac{\partial S(\theta)}{\partial \theta}\boldsymbol{z} + \frac{1}{2|S(\theta)|}\frac{\partial |S(\theta)|}{\partial \theta}. \qquad (16)$$

With a unitary transformation V, the matrix UGU^T can be diagonalised. The diagonal elements are the eigenvalues γ_i. In these coordinates one checks that the determinant $|S(\theta)| = \prod_i 1/(1 + \theta\gamma_i)$ and $\partial|S(\theta)|/\partial\theta = -|S(\theta)| \sum_i \gamma_i/(1 + \theta\gamma_i)$. Furthermore

$$\frac{\partial S(\theta)}{\partial \theta} = -S(\theta)\frac{\partial S(\theta)^{-1}}{\partial \theta}S(\theta) = -S(\theta)UGU^T S(\theta).$$

Thus (16) reads

$$
\begin{aligned}
\Lambda'(\theta) &= \frac{\theta}{2}\boldsymbol{z}^T\left(2S(\theta)-\theta S(\theta)UGU^TS(\theta)\right)\boldsymbol{z}-\frac{1}{2}\sum_i\frac{\gamma_i}{1+\theta\gamma_i}\\
&= \frac{\theta}{2}\boldsymbol{z}^T\left(2S(\theta)-\theta S(\theta)(UGU^T+\mathbf{1}-\mathbf{1})S(\theta)\right)\boldsymbol{z}-\frac{1}{2}\sum_i\frac{\gamma_i}{1+\theta\gamma_i}\\
&= \frac{\theta}{2}\boldsymbol{z}^T\left(2S(\theta)-\theta S(\theta)(S(\theta)^{-1}-\mathbf{1})S(\theta)\right)\boldsymbol{z}-\frac{1}{2}\sum_i\frac{\gamma_i}{1+\theta\gamma_i}\\
&= \frac{\theta}{2}\boldsymbol{z}^T\left(S(\theta)+S(\theta)^2\right)\boldsymbol{z}-\frac{1}{2}\sum_i\frac{\gamma_i}{1+\theta\gamma_i} \qquad (17)
\end{aligned}
$$

Thus equation (3) determining $\overline{\theta}$ reads

$$
\begin{aligned}
k &= \frac{\theta^2}{2}\boldsymbol{z}^T\left(S(\theta)+S(\theta)^2\right)\boldsymbol{z}-\frac{\theta}{2}\sum_i\frac{\gamma_i}{1+\theta\gamma_i}\\
&\quad -\frac{\theta^2}{2}\boldsymbol{z}^TS(\theta)\boldsymbol{z}-\frac{1}{2}\log\prod_i 1/(1+\theta\gamma_i)+\frac{1}{2}\log n\\
&= \frac{1}{2}\left[\theta^2\boldsymbol{l}^TU^TS(\theta)^2U\boldsymbol{l}+\log n-\sum_i\left(\frac{\theta\gamma_i}{1+\theta\gamma_i}-\log(1+\theta\gamma_i)\right)\right]
\end{aligned}
$$

which establishes (6). Since this function is strictly increasing in θ, and goes to infinity as θ goes to its maximal value $\theta_{\max}$, this equation always has a unique positive solution $\overline{\theta}$. ($\theta_{\max}$ is infinity if all eigenvalues γ_i are positive, and it equals $-1/(\min\gamma_i)$ if some γ_i are negative.)

Entering the expressions for $\overline{L}(\boldsymbol{y})$ and $\Lambda(\theta)$ into (4) the worst case scenarios in $\boldsymbol{y}$-coordinates turns out to be a normal with mean $-\overline{\theta}S(\overline{\theta})\boldsymbol{z}$ and covariance matrix $S(\overline{\theta})$. The worst case loss is given by evaluating (17). □

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